Modeling Cyclic RPS Game with Primal-Dual Pair Linear Program

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Abstract

Game theory is the mathematical study of strategic interactions, in which an individual's success depends on his/her own choice as well as the choices of others. The inspection of basic matrix games in two-person zero sum game can be constructed as the mathematical model. The importance of the game is how to find the gamevalue (expected payoff) for each player. This game can be solved by using primal-dual pair linear programming (LP) to obtain the optimal solution is discussed. Linear programming has been shown to be available method for solving zero-sum games. The two-person zero sum games that are well suited for solving using linear programming, rather than suggest the dual solution methodology for the opponent's strategies, and supply a single formulation for both players. This also informs the relationship between the Minimax Theorem of game theory and the Weak Duality Theorem of linear programming. Duality theory is a concept operating on two mathematical programming problems and the possible coincidence of their values.

Keywords: Two person zero-sum game, payoff, Linear programming, Weak duality, Strong duality, Minimax theorem

1. Introduction

Game theory is the branch of mathematics and decision theory concerned with strategic decisions when two or more players complete. The problem of interest involve multiple participants, each of whom has individual strategies related to a common system or shared resources. Because game theory arose from the analysis of competitive scenarios, the problems are called "game" and the participants are called "players". In two person games, each of the players has strategies or courses of action that they might choose. These courses lead to outcomes or payoffs to the decision maker and these payoffs might be any value (positive, negative, or zero). These payoffs are usually presented in a payoff matrix. Linear programming is applied that focuses on solving optimization problems involving linear functions of variables that satisfy linear inequalities. Such problems are important and numerous, including personnel and equipment assignments, optimal use of limited resources, distribution of goods through a transportation network, investment portfolio funding, project or maintenance scheduling, and many more. We construct one type of the game, and use LP duality to give us some insight about behavior in the game and present the two-person game matrices closely and see how one can solve them, that means finding the maximin strategies for the row player, minimax strategies for the column player and the values of the game. To every linear program there is a dual linear program with which it is intimately connected.

2. Preliminaries

2.1 Game

A game is a formal description of a strategic situation.

2.2 Payoff

A payoff is a number also called utility, which reflects the desirability of an outcome to a player, for whatever reason. When the outcome is random, payoffs are usually weighted with their probabilities. The expected payoff in coorporates the player's attitude towards risk.

2.2 Pure Strategies

If a player knows exactly what the other player is going to do, a deterministic situation is obtained and objective function is to maximize the gain. Therefore, the pure strategy is a decision rule always to select a particular course of action.

2.3 Two-person Game

The normal form of two-person game is specified via a pair (R, C) of $m \times n$ payoff matrices. The two players of the game, called the row player and the column player, have respectively m and n pure strategies.

2.4 Two-person Zero-sum Game

A two-person game is said to be zero-sum if $R + C = 0$, that is, $R[i,j] + C[i,j] = 0$ for all $i \in [m]$ and $j \in [n]$.

3. Modeling the Two Person Zero Sum RPS Game[4]

A game with only two players (player I and player II) is called a 'two-person zero-sum game', if the losses of one player are equivalent to the gains of the other so that the sum of their net gains is zero. Two-person zero-sum games are also called rectangular games as these are usually represented by a payoff matrix in a rectangular form.

The fundamental example of the two-person zero-sum game is **Rock-Paper-Scissors** (RPS) game. The RPS game is also a basic model system for studying decision-making of human subjects in competitive environments and the associated social dynamics and non-equilibrium physics.

In this game, Player I (P1) and Player II (P2) simultaneously display one of the three objects: rock, paper or scissors. In this game, the players face each other and simultaneously display their hands in one of the three following shapes: a fist denoting a rock, the forefinger and middle finger extended and spread so as to suggest scissors, or a downward facing palm denoting a sheet of paper.

If the players both choose the same object to display, there is no payoff. If they choose different objects, then scissors win over paper (scissors cut paper), rock wins over scissors (rock breaks scissors), and paper wins over rock (paper covers rock).

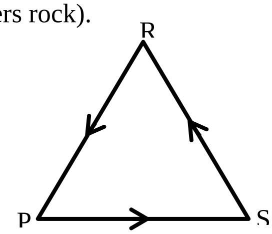

Figure 1. The Rock-Paper-Scissors game (Relation R breaks S, P wins over R, S cuts P among the three actions).

The RPS game is a special case of a "cyclic" game.

Next, the two-person zero sum game with $m \times n$ game matrix is of the form[2]

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Suppose the row player uses strategy

$$x = (x_1, \dots, x_m) \in P^m.$$

Then the column player would use his jth strategy such that

$$a_{1j}p_1 + a_{2j}p_2 + \cdots + a_{mj}p_m$$

is minimum among $j = 1,2,\cdots,n$. Thus the payoff to the row player that he can guarantee is

$$\min_{j=1.2,\dots,n} \{a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_j\}$$

Hence if the above expression attains its maximum at $x = p \in P^m$, then p is a maximin strategy for the row player. Moreover, the value of the game is

$$z = \max_{x \in P^m} \min_{j=1.2,\dots,n} \{a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_j\}$$

By introducing a new variable z, we can restate the maximin problem, that is finding a maximin strategy, as the following linear programming problem

$$\begin{aligned} &\max \quad z \\ &\textit{suject to} \quad a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \geq z \\ &\qquad a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \geq z \\ &\qquad \vdots \\ &\qquad a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \geq z \\ &\qquad p_1 + p_2 + \cdots + p_m = 1 \\ &\qquad p_1, p_2, \dots, p_m \geq 0 \end{aligned}$$

Similarly, to find a minimax strategy for the column player, we need to solve the following minimax problem

$$\begin{aligned} &\min \quad w \\ &\textit{suject to} \quad a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n \leq w \\ &\qquad a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n \leq w \\ &\qquad \vdots \\ &\qquad a_{m1}q_1 + a_{m2}q_2 + \cdots + a_{mn}q_n \leq w \\ &\qquad q_1 + q_2 + \cdots + q_n = 1 \\ &\qquad q_1, q_2, \dots, q_n \geq 0 \end{aligned}$$

To solve the maximin and minimax problems, first we transform them to a pair of primal and dual problems.

If the payoff upon winning or losing is one unit, then the matrix of the game is as follows:

$$\begin{array}{cc} & \text{Player I} \\ & \begin{array}{ccc} R & P & S \end{array} \\ \text{Player II} \begin{array}{c} R \\ P \\ S \end{array} & \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array}$$

The above matrix is payoff for *column player* P1[4].

First, we want to find minimax solution for P1 of the above payoff matrix.

Let *r*, *p*, *s* be probability of P2 playing rock, paper, scissors respectively.

The expected payoffs for P1 given different actions of P2 are as follows:

P2 plays rock: $z = r(0) + p(1) + s(-1) = p - s$

P2 plays paper: $z = r(-1) + p(0) + s(1) = -r + s$

P2 plays scissors: $z = r(1) + p(-1) + s(0) = r - p$.

Again, P2 is trying to minimize P1's utility, so we have

$$z \leq p - s;$$
$$z \leq -r + s;$$
$$z \leq r - p.$$

Now we want to find the solution of *r*, *p*, *s*.

$$\left.\begin{array}{lr} \text{maximize} & z \\ \text{subject to} & p - s \geq z \\ & -r + s \geq z \\ & r - p \geq z \\ & r + p + s = 1. \\ & r \geq 0, p \geq 0, s \geq 0. \end{array}\right\} \quad (1)$$

Expression (1) is called a mathematical programming problem.

Next, we want to present the linear programming (LP).

4. Linear Programming

4.1 Definitions

4.1.1 Linear programming [3]

Linear Programming (LP) might best be called Linear Optimization: it means finding maxima and minima of linear functions of several variables subject to constraints that are linear equations or linear inequalities.

The general form of LP problem is

$$\begin{array}{ll} \text{minimize} & c.x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

where $A \in R^{m\times n}$, $b \in R^m, c, x \in R^n$.

We also call $c.x$ the **objective function**, and $Ax \geq b$ **the constraints**.

When both the objective and all the constraints in Expression (1) are linear functions, then the optimization problem is called a linear programming problem.

The word “programming” has the old-fashioned meaning of “planning” and was chosen in the forties, before the advent of computers.

4.1.2 Primal and its dual[3]

For a typical presentation of every linear primal problem:

$$\begin{array}{ll} \text{minimize} & c.x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

the dual linear program

$$\begin{array}{ll} \text{maximize} & b.y \\ \text{subject to} & A^T y \leq c \\ & y \geq 0. \end{array}$$

4.1.3 Weak duality[3]

If x is any feasible solution of the primal and y is any feasible solution of the dual, then $c.x \leq b.y$.

Moreover, if equality holds, that is if $c.x = b.y$, then x is be an optimal solution of the primal LP and y is an optimal solution of the dual LP.

4.1.4 Strong duality[3]

Assume a primal-dual pair of linear problems satisfies any of the following conditions:

i. the primal has a optimal solution,
ii. the dual has an optimal solution, or,
iii. both primal and dual are feasible.

Then both primal and dual must have optimal solutions (say $x^* and\ y^*$) and the optimal values of the objective functions are equal

($c.x^* = b.y^*$ using the above notation for the primal and dual).

4.2 Minimax theorem[2]

Let A be an $m \times n$ game matrix. Then there exists real number $v \in R$, mixed strategy for the row player $p \in R^m$, and mixed strategy for the column player $q \in R^n$ such that

$i.\ pAy^T \geq v, for\ any\ y \in P^n$
$ii.\ xAq^T \leq v, for\ any\ x \in P^m$
$iii.\ pAq^T = v$.

In the above theorem, the real number $v = v(A)$ is called the value, or the security level, of the game matrix A. The strategy p is called a *maximin strategy* for the row player and the strategy q is called a *minimax strategy* for the column player. The value v of a matrix is unique. However maximin strategy and minimax strategy are in general not unique.

5. Solving the Mathematical Model by Linear Programming

a. Let $x_1 = r, x_2 = p, x_3 = s$ be the probability that P2 chooses action for {Rock, Paper, Scissors}.

Then P2's maximin strategy can be found with the following optimization model:

$$\begin{aligned} &\text{maximize} \quad \min\{x_2 - x_3, -x_1 + x_3, x_1 - x_2\} \\ &\text{subject to} \quad x_1 + x_2 + x_3 = 1 \\ &\qquad\qquad\quad x_1 \geq 0,\ x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

The above model is equivalent to the following linear program:

$$\begin{aligned} &\text{maximize} && z \\ &\text{subject to} && x_2 - x_3 \geq z \\ &&& -x_1 + x_3 \geq z \\ &&& x_1 - x_2 \geq z \\ &&& x_1 + x_2 + x_3 = 1 \\ &&& x_1 \geq 0,\ x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

b. Let v_i be the probability the P1 chooses action i , $i = 1, 2, 3$ for $i \in$ {Rock, Paper, Scissors}.

Then P1 minimax strategy can be found with the following optimization model:

$$\begin{aligned} &\text{minimize } \max\{-v_2 + v_3, v_1 - v_3, -v_1 + v_2\} \\ &\text{subject to} \qquad v_1 + v_2 + v_3 = 1 \\ &\qquad\qquad\qquad\quad v_1 \geq 0,\ v_2 \geq 0, v_3 \geq 0. \end{aligned}$$

This model is equivalent to the following linear program:

$$\begin{aligned} &\text{minimize} && w \\ &\text{subject to} && -v_2 + v_3 \leq w \\ &&& v_1 - v_3 \leq w \\ &&& -v_1 + v_2 \leq w \\ &&& v_1 + v_2 + v_3 = 1 \\ &&& v_1 \geq 0,\ v_2 \geq 0, v_3 \geq 0 \end{aligned}$$

c. Let's rewrite P1's linear program so that all the decision variables are on the left hand side of the constraints, and all the constants are on the right hand side:

$$\begin{aligned} &\text{maximize} && z \\ &\text{subject to} && z - x_2 + x_3 \leq 0 \\ &&& z + x_1 - x_3 \leq 0 \\ &&& z - x_1 + x_2 \leq 0 \\ &&& x_1 + x_2 + x_3 = 1 \\ &&& x_1 \geq 0,\ x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

The relation between the primal and dual is

- ➢ Nonnegative variables in primal ⇔ inequality constraints in dual
- ➢ Free variables in primal ⇔ equality constraints in dual

Table 1. Dual problem for primal of P2

$\min w$	z $free$	$x_1 \geq 0$	$x_2 \geq 0$	$x_3 \geq 0$	
$v_1 \geq 0$	1	0	-1	1	≤ 0
$v_2 \geq 0$	1	1	0	-1	≤ 0
$v_3 \geq 0$	1	-1	1	0	≤ 0
w $free$	0	1	1	1	$= 1$
	$= 1$	≥ 0	≥ 0	≥ 0	

The dual of the above primal program is

$$\begin{aligned} \text{minimize} \quad & w \\ \text{subject to} \quad & v_1 + v_2 + v_3 = 1 \\ & v_2 - v_3 + w \geq 0 \\ & -v_1 + v_3 + w \geq 0 \\ & v_1 - v_2 + w \geq 0 \\ & v_1 \geq 0,\ v_2 \geq 0, v_3 \geq 0;\ w \text{ free.} \end{aligned}$$

We can see that the dual of primal for P2 is equivalent to the primal for P1.

The primal problem in inequality form so that the primal and dual problems are symmetric. Thus, any statement that is made about the primal problem immediately has an analog for the dual problem, and conversely.

d. We can represent the "play each alternative with probability $1/3$" strategy for player II with the solution

$$x_1 = x_2 = x_3 = \tfrac{1}{3}.$$

Note that this solution is feasible for player II's linear program. The objective function value of this feasible solution is

$$\min\ \{\tfrac{1}{3} - \tfrac{1}{3}, -\tfrac{1}{3} + \tfrac{1}{3}, \tfrac{1}{3} - \tfrac{1}{3}\} = 0.$$

e. Similarly, we can represent the "play each alternative with probability $1/3$" strategy for player I with the solution

$$v_1 = v_2 = v_3 = \tfrac{1}{3},$$

which is feasible for player I's linear program. The objective function value of this feasible solution is

$$\max\ \{-\tfrac{1}{3} + \tfrac{1}{3}, \tfrac{1}{3} - \tfrac{1}{3}, -\tfrac{1}{3} + \tfrac{1}{3}\} = 0.$$

f. We have feasible solutions to P1's and P2's linear programs with equal objective function values. Since these linear programs form a primal-dual pair, by weak duality, these solutions must be optimal. Therefore, we can conclude that the "play each alternative with probability $1/3$" strategy is optimal for both P1 and P2.

Conclusion

Duality theory is important for developing computational procedures that take advantage of the relationships between the primal and dual problems. Duality theory is closely related to game theory, and indicate the basic relationships. Moreover, the two-person *constant-sum* game, *n-person* zero-sum game and *n-person constant-sum game* can be solved by using the Duality Theorem.

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